Why do we live in a Riemannian space-time ?

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Abstract

We start from the pure Einstein-Hilbert action $S = \int \lambda^2 R \star 1$ in Metric-Affine-Gravity, with the orthonormal metric $g_{ab} = \eta_{ab}$. We get an effective Levi-Civita Dilaton gravity theory in which the Dilaton field is related to the scaling of the gravitational coupling.
When the Weyl symmetry is broken the resulting Einstein-Hilbert term is equivalent to the Levi-Civita one, using the projective invariance of the model, the non-metricity and torsion may be removed, so that we get a theory perfectly equivalent to General Relativity. This may explain why low energy gravity is described by a Riemannian connection.

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Among the four fundamental interactions, the two feeble are characterised by dimensional coupling constants, $G_F = (300Gev)^{-2}$ and Newton's coupling constant $G_N = (10^{19}Gev)^{-2}$.
It is well known that interactions with dimensional coupling constants present many problems among which there is the renormalizability.
The success of the Weinberg-Salam model has told us that the weak interaction is characterised by a dimensionless coupling constant and the dimensions of G_F are due to the spontaneous symmetry breaking mechanism, so that $G_F \cong \frac{1}{v_W{}^2}$ where $v_W \cong 300Gev$ is the vacuum expectation value of the Higgs field.
The weakness of the weak interaction being related to the large vacuum expectation value of the scalar field [1].
It is believed that similar mechanisms may occur for gravity, which is characterised by a dimensionless coupling constant ξ. The weakness of gravity then would be related to the symmetry breaking at very high energies [2-4]. This may be obtained starting from a Dilaton theory which presents Weyl scale invariance. The potential $V(\psi)$ which appears in the action is assumed to have its minimum at $\psi = \sigma$, then when $\psi = \sigma$ the Dilaton theory reduces to the Einstein-Hilbert action with gravitational constant $G_N = \frac{1}{8\pi\xi\sigma^2}$.
It has been shown that in the context of Metric-Affine-Gravity [5] the kinetic term for the dilaton may be obtained from a generalised Einstein-Hilbert term [6].
In this letter we continue the analysis of the model considerd in [6].
We investigate in the Tucker-Wang approach to non Riemannian gravity [6] the simple action:

$$S = \int \lambda^2 R \star 1 \tag{1}$$

Where R is the scalar curvature associated with the full non Riemannian connection.
In the Tucker-Wang approach to MAG [7] we choose the metric to be orthonormal $g_{ab} = \eta_{ab} = (-1, 1, 1, 1, ...)$ and we vary with respect to the coframe e^a and the connection $\omega^a{}_b$ considered as independent gauge potentials.
We will study the two different cases where λ is a dynamical variable subjected to aWeyl rescaling or the case when it is a constant.
We will prove that when the Weyl invariance is broken the theory obtained from (1) is perfectly equivalent to General Relativity. The breaking of Weyl

symmetry may then give a giustification of why the low energy limit of gravity is Riemmanian.
Before going into the calculations let us define how the Weyl rescaling transformations are defined in the Tucker-Wang approach.
Since the metric g_{ab} is fixed we act only on the variable λ and the coframe e^a [1].
The Einstein-Hilbert term can be written in the form:

$$R \star 1 = R^a{}_b \wedge \star(e_a \wedge e^b) \tag{2}$$

Since the curvature two forms depends on the connection $\omega^a{}_b$, but not on the coframe it will not be affected by a rescaling of the coframe. The term $\star(e_a \wedge e^b)$ is a $n-2$ form. Then it is easy to see that if we introduce the rescaling defined by:

$$\begin{aligned} \lambda &\rightarrow \lambda\, \Omega^p \\ e^a &\rightarrow e^a\, \Omega^q \end{aligned} \tag{3}$$

The scale invariance of (1) holds if we satisfy the condition:

$$2\,p = -(n-2)\,q \tag{4}$$

In what follows we will suppose (4) to hold.

If we consider the connection variation of (1) we get:

$$D \star (e_a \wedge e^b) = -\frac{2}{\lambda}[d\lambda \wedge \star(e_a \wedge e^b)] = A(\lambda)[d\lambda \wedge \star(e_a \wedge e^b)] \tag{5}$$

with $A(\lambda) = -\frac{2}{\lambda}$.

The full non Riemannian Einstein Hilbert term can be written as:

$$\begin{aligned} R \star 1 = \overset{o}{R} \star 1 - \hat{\lambda}^a{}_c \wedge \hat{\lambda}^c{}_b \wedge \star(e^b \wedge e_a) - \\ d(\hat{\lambda}^a{}_b \wedge \star(e^b \wedge e_a)) \end{aligned} \tag{6}$$

[1] We could have introduced a rescaling for the connection too, but in here we are looking for the simplest model so we consider the connection scale invariant

where $\hat{\lambda}^a{}_b$ is the traceless part of the non Riemannian part of the connection $\lambda^a{}_b$.

By considering the coframe variation we get then the generalized Einstein equations:

$$\begin{aligned}\lambda^2 \overset{o}{R}{}^a{}_b \wedge \star(e_a \wedge e^b \wedge e_c) - 2\lambda[\hat{\lambda}^a{}_b \wedge d\lambda \wedge \star(e^b \wedge e_a \wedge e_c)] \\ +\lambda^2[\hat{\lambda}^a{}_d \wedge \hat{\lambda}^d{}_b] \wedge \star(e_a \wedge e^b \wedge e_c) = 0\end{aligned} \tag{7}$$

The Cartan equation can be written as:

$$D \star (e^a \wedge e_b) = A(\lambda)[d\lambda \wedge \star(e^a \wedge e_b)] = F^a{}_b \tag{8}$$

We get from (8):

$$f_{cab} = A(\psi) i_c(\star(d\lambda \wedge \star(e_a \wedge e_b))) \tag{9}$$

the solution of which gives for the traceless part of the non-metricity and torsion:

$$\begin{aligned}\hat{Q}^{ab} &= 0 \\ \hat{T}_c &= 0\end{aligned} \tag{10}$$

and

$$T = \frac{n-1}{2n} Q + \frac{1-n}{n-2} A(\lambda) d\lambda \tag{11}$$

the solution for the nonmetricity and torsion can then be written as:

$$\begin{aligned}Q_{ab} &= \frac{1}{n} g_{ab} Q \\ T^a &= \frac{1}{2n}(e^a \wedge Q) - \frac{1}{n-2}(e^a \wedge d\lambda) A(\lambda)\end{aligned} \tag{12}$$

Using the expression of $\lambda^a{}_b$ as a function of T^a and Q_{ab} [7]:

$$2\lambda_{ab} = i_a T_b - i_b T_a - (i_a i_b T_c + i_b Q_{ac} - i_a Q_{bc}) e^c - Q_{ab} \tag{13}$$

we obtain:

$$\lambda_{ab} = -\frac{1}{2n} g_{ab} Q + \frac{1}{n-2} A(\lambda)(i_a(d\lambda)e_b - i_b(d\lambda)e_a) \tag{14}$$

and the traceless part:

$$\hat{\lambda}_{ab} = \frac{1}{n-2} A(\lambda)(i_a(d\lambda)e_b - i_b(d\lambda)e_a) \tag{15}$$

By using the previous expression in the generalised Einstein equations we get after some calculations:

$$\lambda^2 \overset{o}{G}_c - \beta[d\lambda \wedge i_c \star d\lambda + i_c d\lambda \wedge \star d\lambda] = 0 \tag{16}$$

where $\overset{o}{G}_c = \overset{o}{R}{}^a{}_b \wedge \star(e_a \wedge e^b \wedge e_c)$ and $\beta = 4\frac{n-1}{n-2}$ and the superscript (o) refers to the Levi-Civita part.

The variation of (1) with respect to λ gives:

$$\lambda\, R \star 1 = 0 \tag{17}$$

Which can be shown to be equivalent to:

$$\lambda^2 \;\overset{o}{R} \star 1 - 4\frac{n-1}{n-2}(d\lambda \wedge \star d\lambda) = 0 \tag{18}$$

In conclusion we get the equations:

$$\begin{aligned} \lambda^2 \overset{o}{G}_c - 4\frac{n-1}{n-2}[d\lambda \wedge i_c \star d\lambda + i_c d\lambda \wedge \star d\lambda] = 0 \\ +\lambda \,\overset{o}{R} \star 1 - \frac{4}{\lambda}\frac{n-1}{n-2}(d\lambda \wedge \star d\lambda) = 0 \end{aligned} \tag{19}$$

For $n = 4$ we have:

$$\begin{aligned} \lambda^2 \overset{o}{G}_c - 6k[d\lambda \wedge i_c \star d\lambda + i_c d\lambda \wedge \star d\lambda] = 0 \\ +\lambda \,\overset{o}{R} \star 1 - \frac{6}{\lambda}(d\lambda \wedge \star d\lambda) = 0 \end{aligned} \tag{20}$$

The Einstein equations in (19) coincide with the conformally invariant Einstein equations obtained starting from the action:

$$S = \int \lambda^2 \overset{o}{R} \star 1 + 4\frac{n-1}{n-2}(d\lambda \wedge \star d\lambda) \tag{21}$$

We have to remember however that this equivalence holds with the amendment that the Weyl rescaling is defined for the coframe and not for the metric since g_{ab} is fixed to be orthonormal.
What has been said is valid because we are assuming that λ may be affected by a Weyl rescaling.
Suppose now to consider the situation in which the Weyl symmetry is broken, we choose then a certain value of λ :

$$\lambda = \lambda_0 \tag{22}$$

The Cartan equation then reduces to:

$$D \star (e_a \wedge e^b) = 0 \tag{23}$$

The solution of which is:

$$\begin{aligned} Q_{ab} &= \frac{1}{n} g_{ab} Q \\ T^a &= \frac{1}{n-1}(e^a \wedge T) \\ T &= \frac{n-1}{2n} Q \\ \lambda_{ab} &= -\frac{1}{2n} g_{ab} Q \end{aligned} \tag{24}$$

But what is more important is that:

$$\hat{\lambda}_{ab} = 0 \tag{25}$$

so that we get:

$$\begin{aligned} R \star 1 &\equiv \overset{o}{R} \star 1 \\ G_c &= \overset{o}{G}_c \end{aligned} \tag{26}$$

So the action and the field equations become equivalent to the Einstein theory obtained from the action:

$$S = \int \lambda_o{}^2 \overset{o}{R} \star 1 \tag{27}$$

That is we get the vacuum Einstein equations:

$$\lambda_o{}^2 G_c = 0 \tag{28}$$

with torsion and non-metricity given by (24).

In conclusion starting from the action $S = \int \lambda^2 R \star 1$ in MAG we are able to exhibit two theories depending on whether λ is a constant or a Weyl field variable. In the latter case we obtain a Dilaton-Levi-Civita model for the Einstein sector, in which Q_{ab}, T^a, T are given by (11-12). In the former we get a vacuum Einstein theory with non-metricity and torsion (parametrised by Q) and related by (26).

The next step is to enquiry why Q_{ab} and T^a are zero in the broken phase.

To prove that we invoke the projective invariance [12,13] of action (1). Indeed action (1) is invariant under the projective transformation:

$$\omega^a{}_b \to \omega^a{}_b + \delta^a{}_b P \tag{29}$$

with P arbitrary 1-form.

Under this transformation Q and T transforms as:

$$\begin{aligned} Q &\to Q - 2n\, P \\ T &\to T + (1-n)\, P \end{aligned} \tag{30}$$

If I choose $P = \frac{Q}{2n}$ then I get $Q' = 0$ and:

$$T' = T + (1-n)\, P = T + \frac{1-n}{2n} Q \tag{31}$$

Which vanishes on account of the third of (24).

The conclusion is that the non-Riemmanian fields can be removed using a projective transformation and we are left with a theory completely equivalent to General Relativity.

The interesting issue of studying the stability against perturbation around

$Q = T = 0$ will be considered in another paper.

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